Short Exam 1

Problem. “Stationary” orbit. A small body can slide without friction along the inner surface of a fixed sphere of radius R. The position of the body is described by the angle θ between its radius vector and the vertical line passing through the centre, with $\theta = 0$ corresponding to the lowest point of the sphere. The acceleration due to gravity is g.

(a) The body is placed at a point with a deviation angle θ_0 from the vertical, as shown on Figure 1 ($\theta_0 < \pi/2$). What initial horizontal velocity v_0 must be imparted to the body so that it continues to move along a horizontal circle, i.e. along the parallel through the initial point? **(1.5 pt)**

(b) Suppose that at the initial moment, in addition to the horizontal velocity $\vec{v}_0$ determined in part (a), the body is also given some small extra velocity $\vec{v}_1$ ($v_1 \ll v_0$) in the direction of the meridian on which it is placed. For what initial angle θ_0 will the body return to the starting point after exactly one lap, with a speed identical in magnitude to the initial speed? **(3.5 pt)**

Time: 45 minutes

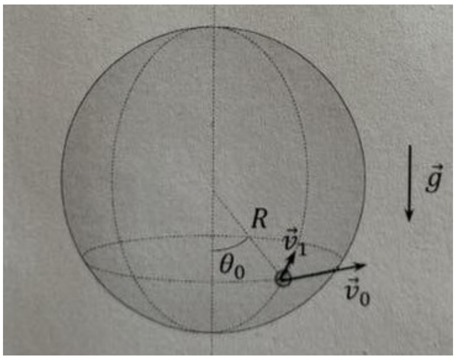

Figure 1

Solution. (a) The body moves in a horizontal circle of radius $R\sin\theta_0$. The normal force N from the sphere is directed toward the centre of the sphere. Resolving the forces vertically and centripetally:

$$N\cos\theta_0 = mg,$$

$$N\sin\theta_0 = \frac{mv_0^2}{R\sin\theta_0}.$$

Dividing these two equations gives us $\tan\theta_0 = v_0^2/(gR\sin\theta_0)$. Hence

$$\boxed{v_0 = \sqrt{gR\sin\theta_0\tan\theta_0}.}$$

(b) Before we give the body a kick, the angular velocity on its circular orbit is

$$\omega_0 = \frac{v_0}{R\sin\theta_0} = \sqrt{\frac{g}{R\cos\theta_0}}.$$

With the extra kick v_1, the body no longer stays on the parallel but instead oscillates around it. Since the normal force and the gravity force do not exert a torque about the z-axis, the angular momentum about it conserved. Denoting the rate of the azimuthal angle by $\dot{\phi}$, we can then write

$$L_z = mR^2 \sin^2\theta \, \dot{\phi} = \text{const} = mR^2 \sin^2\theta_0 \, \omega_0,$$

Next, we need an equation of motion along the meridian. From force balance, we have

$$N\cos\theta - mg = m\ddot{z},$$
$$N\sin\theta = m\dot{\phi}^2 R\sin\theta$$

After plugging in the kinematic relation $\ddot{z}\cos\theta = R\ddot{\theta}$, we reach

$$R\ddot{\theta} = R\sin\theta\cos\theta\,\dot{\phi}^2 - g\sin\theta.$$

We want to solve this for small deviations from θ_0. In the above equations, let $\theta = \theta_0 + \eta$, where $\eta \ll 1$. Our goal is to expand to first order in η. Using $\sin(\theta_0+\eta) \approx \sin\theta_0 + \eta\cos\theta_0$ and $\cos(\theta_0+\eta) \approx \cos\theta_0 - \eta\sin\theta_0$ in the angular momentum conservation statement, we obtain

$$\dot{\phi} = \frac{\omega_0 \sin^2\theta_0}{\sin^2(\theta_0+\eta)} \approx \omega_0\left(1 - \frac{2\eta\cos\theta_0}{\sin\theta_0}\right).$$

After substituting this in the equation for $\ddot{\theta}$, we get

$$\ddot{\eta} = -\omega_0^2(1+3\cos^2\theta_0)\,\eta.$$

This is simple harmonic motion with angular frequency

$$\Omega = \omega_0\sqrt{1+3\cos^2\theta_0}.$$

Since $\Omega > \omega_0$, the body oscillates in θ faster than it revolves in ϕ. For the body to cross the starting point after one full revolution in ϕ, it must have completed $\{3/2, 2, 5/2, \dots\}$ periods in θ. Since the factor $\sqrt{1+3\cos^2\theta_0}$ ranges from 1 to 2, the only nontrivial option is $\Omega = (3/2)\omega_0$. This corresponds to $\cos^2\theta_0 = 5/12$, so $\boxed{\theta_0 \approx 49.8^\circ.}$

Short Exam 2

Problem. Transparent sphere. A transparent sphere of refractive index n has radius R. Inside it, at a distance r from its centre, there is a point source of light.

(a) What fraction k of the light beam emitted by the source will undergo total internal reflection? What are the constraints on r for total internal reflection to be observed?

(b) A spherical screen of radius L is placed around the sphere. The sphere and the screen are concentric. What must L be so that a bright glowing point can be observed on the screen? What are the constraints on n and r for this to occur?

Time: 50 minutes

Solution. (a) Consider a ray emitted from the source S making an angle β with the radius OS. Let the ray strike the surface at point P with an angle of incidence α. Using the sine rule in $\triangle OSP$:

$$\frac{\sin\alpha}{r} = \frac{\sin\beta}{R} \quad \Rightarrow \quad \sin\alpha = \frac{r}{R}\sin\beta.$$

Total internal reflection (TIR) occurs when $\sin\alpha \geq 1/n$. The maximum value of $\sin\alpha$ is r/R (for $\beta = 90^\circ$). Hence, to get any TIR at all, we need $r/R \geq 1/n$, or $\boxed{r \geq R/n.}$

The condition $\sin\alpha = 1/n$ defines a critical emission angle β_c such that $\sin\beta_c = R/(nr)$. Rays with $\beta_c < \beta < \pi - \beta_c$ will undergo TIR. The solid angle that corresponds to this volume can be found by taking all space and subtracting off two spherical caps of half-angle β_c. Thus, the solid angle is $\Omega = 4\pi - 2 \cdot 2\pi(1 - \cos\beta_c) = 4\pi \cos\beta_c$. The fraction is

$$k = \cos\beta_c = \boxed{\sqrt{1 - \left(\frac{R}{nr}\right)^2}.}$$

(b) A bright point implies a focus. This occurs when paraxial rays from S converge at L after one refraction. We will use the formula for refraction at a spherical boundary:

$$\frac{n_1}{s} + \frac{n_2}{v} = \frac{n_2 - n_1}{R'},$$

where $n_{1,2}$ are the refractive indices on each side, s is the object distance, v is the image distance, and R' is the radius of curvature. If you're unfamiliar with this, the derivation proceeds similarly to that of the lensmaker's equation – simply draw the boundary and consider two rays from the object, one central and one that refracts at a small angle. The image distance will correspond to their intersection point.

Back to our problem, to get a real image outside, we need a symmetrical setup, so the rays should either pass through the "far" side ($s = R + r$) or the "near" side ($s = R - r$). Let's cover the first case. Using $s = R + r$, $n_1 = n$, $n_2 = 1$, and $R' = -R$ (concave boundary):

$$\frac{n}{R+r} + \frac{1}{v} = \frac{1-n}{-R} \quad \Rightarrow \quad \frac{1}{v} = \frac{r(n-1) - R}{R(R+r)}.$$

For a real image, we need $v > 0$, so $\boxed{r > R/(n-1)}$. Since $r < R$, this also implies $\boxed{n > 2}$. Finally, the distance from the center is $L = v + R$:

$$L = R + \frac{R(R+r)}{r(n-1) - R} = \boxed{\frac{nrR}{r(n-1) - R}.}$$

For the second case, we have

$$\frac{n}{R-r} + \frac{1}{v} = \frac{1-n}{-R} \quad \Rightarrow \quad \frac{1}{v} = \frac{(1-n)r - R}{R(R-r)}.$$

This is always negative, so we can't get an image on the screen this way. We're left with a single answer only.

Short Exam 3

Problem. Diamagnetism. Consider an electron moving uniformly in a circle (for example, on a circular orbit around the stationary nucleus in a hydrogen atom). This system has a magnetic dipole moment $\mu = IS$, where I is the current associated with the motion of the electron along the circle, and S is the area of the circle bounded by this orbit.

(a) Find the relationship between the magnetic dipole moment and the angular momentum of the electron.

The system is placed in a magnetic field B perpendicular to the plane of the circle.

(b) Find the change in the magnetic moment when the magnetic field changes from 0 to B. Assume that the radius of the orbit does not change.

In the formula you've obtained, the radius of the circle appears as ρ^2, whereas the quantum mechanical formula involves the expectation value $\langle \rho^2 \rangle$ instead. The ground state of the hydrogen atom is spherically symmetric, and therefore the following relation holds:

$$\langle \rho^2 \rangle = \frac{2}{3} \langle r^2 \rangle,$$

where r is the distance from the electron to the nucleus in three-dimensional space. The magnetic moment of one mole of matter can be obtained by summing over all electrons in a given atom and then over all atoms. The ratio of the magnetic moment ΔM to the magnetic induction B is called the magnetic susceptibility χ, with

$$\chi = \mu_0 \frac{\Delta M}{B}.$$

(c) Using the experimental value of the molar magnetic susceptibility of helium in the gaseous state ($\chi = -2.36 \times 10^{-11}\,\mathrm{m^3/mol}$), estimate the root-mean-square distance between the electron and the nucleus in the ground state of the helium atom.

Time: 60 minutes

Solution. (a) The current due to the electron is $I = e/T = ev/(2\pi\rho)$, while the area of its orbit is $S = \pi\rho^2$, and so the magnetic moment is $\mu = ev\rho/2$. The angular momentum is $L = mv\rho$. We find that the ratio of the magnetic moment to the angular momentum is a constant:

$$\boxed{\frac{\mu}{L} = \frac{e}{2m}.}$$

This constant is known as the gyromagnetic ratio.

(b) The change of the magnetic moment can be expressed as $\Delta\mu = e\Delta v\rho/2$, where Δv is the change in the orbital speed due to the additional Lorentz force after applying B. Assume that Δv is small. The magnetic force is then $e(v + \Delta v)B \approx evB$. The Coulomb attractive force remains the same, so we equate the magnetic force to the change in the centripetal force $\Delta(mv^2/\rho) \approx 2mv\Delta v/\rho$. Thus $\Delta v = eB\rho/2m$. The change in the magnetic moment has magnitude

$$\boxed{\Delta\mu = \frac{e^2\rho^2}{4m} B.}$$

It is always directed opposite to B as per Lenz's law.

(c) The problem statement is somewhat unclear. By inspecting the units of χ, we see that ΔM is the magnetic moment per mole of matter. For helium (2 electrons per atom), we have $\Delta M = 2N_A\langle\Delta\mu\rangle$. Now we substitute this into the expression for χ, accounting for the fact that $\Delta\mu$ and B are opposite:

$$\chi = \mu_0 \frac{\Delta M}{B} = -\frac{\mu_0 N_A e^2 \langle \rho^2 \rangle}{2m} = -\frac{\mu_0 N_A e^2 \langle r^2 \rangle}{3m}.$$

Solving for the RMS distance $r_{\mathrm{rms}} = \sqrt{\langle r^2 \rangle}$:

$$r_{\mathrm{rms}} = \sqrt{-\frac{3m\chi}{\mu_0 N_A e^2}} = \boxed{5.8 \times 10^{-11}\,\mathrm{m}.}$$

Theoretical Exam

Problem 1. Channel. A channel with a rectangular cross-section is confined by a concrete slab of height $a = 2.0\,\mathrm{m}$ and width $b = 0.2\,\mathrm{m}$, as shown on Figure 2. The coefficient of friction between the slab and the bottom of the channel is $k = 1$. The density of concrete is $\rho_1 = 2.3 \times 10^3\,\mathrm{kg/m^3}$ and the density of water is $\rho_0 = 1.0 \times 10^3\,\mathrm{kg/m^3}$. Find the maximum water level H that the slab can hold.

Solution. The slab can fail in two ways: it can either slide horizontally or tip over (rotate) about its outer bottom edge. We will analyze the forces acting on a length L along the channel. First, consider the condition for sliding. The horizontal force exerted by the water is $F_H = \int_0^H \rho_0 g L y\,\mathrm{d}y = \frac{1}{2}\rho_0 g L H^2$. The weight of the concrete slab is $W = \rho_1 g L a b$. The maximum static friction force is $F_f = kW$. So, for the slab not to slide, we require

$$\frac{1}{2}\rho_0 g L H^2 \le k\rho_1 g L a b \quad \Rightarrow \quad H \le \sqrt{\frac{2k\rho_1 a b}{\rho_0}} = 1.36\,\mathrm{m}.$$

Next, consider the condition for tipping about the outer bottom corner. The torque due to the water is $\tau_w = \int_0^H \rho_0 g L y (H - y)\,\mathrm{d}y = \frac{1}{6}\rho_0 g H^3$. The restoring torque due to gravity is $\tau_g = \frac{1}{2}Wb = \frac{1}{2}\rho_1 g a b^2$. For the slab not to tip, we require:

$$\frac{1}{6}\rho_0 g H^3 \le \frac{1}{2}\rho_1 g a b^2 \quad \Rightarrow \quad H \le \sqrt[3]{\frac{3\rho_1 a b^2}{\rho_0}} = 0.82\,\mathrm{m}.$$

Comparing the two limits, we see that the slab will tip over long before it starts sliding. Thus, the maximum water level is $\boxed{H = 0.82\,\mathrm{m}.}$

Problem 2. Oscillating rods. Two identical uniform rods, each of mass m and length l, are connected by a hinge so that they can rotate relative to each other without any friction. The free ends of the rods are connected by a spring with constant k, such that at equilibrium the rods form a right angle (Figure 3). The system is in weightlessness.

The rods are displaced in opposite directions from their equilibrium position by a small angle, and then are released from rest. Find an expression for the frequency ν of the resulting oscillations of the system.

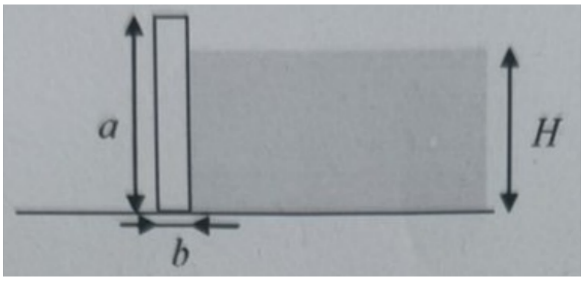

Figure 2

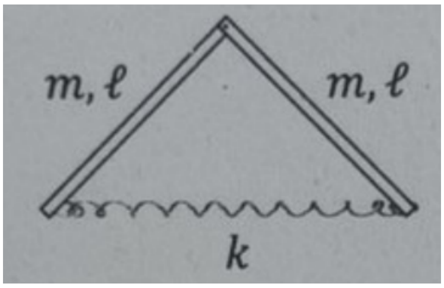

Figure 3

Solution. Since there are no external forces acting on the system, the centre of mass remains stationary. Decompose the motion of the rods into translational motion of their centres plus rotational motion around the centres. Looking at the figure, the vertical components of the translational velocities of the rods must be equal. However, the centre of mass stays put, so these components are zero – the centres of the rods move only horizonally.

We'll solve the problem using energy methods, expressing everything through the angle θ that the rods make with the vertical. Denote the deviation from the initial angle $\theta_0 = 45°$ by α, such that $\alpha = \theta - \theta_0$.

Let's start with the potential energy of the spring. The distance between the free ends is $d = 2l\sin\theta$. In terms of α, this is

$$d = 2l\sin\left(\frac{\pi}{4}+\alpha\right) = 2l\left(\sin\frac{\pi}{4}\cos\alpha + \cos\frac{\pi}{4}\sin\alpha\right) \approx l\sqrt{2}(1+\alpha),$$

so the extension is $\Delta x = d - \sqrt{2}l = \sqrt{2}l\alpha$ and the potential energy is $U = k(\Delta x)^2/2 = kl^2\alpha^2$.

Next, let's find the translational kinetic energy of the rods K_t. The distance between their centres is always $l\sin\theta$ (one half of the distance between the free ends). The velocity of each is then $l\cos\theta\,\dot{\alpha}/2$. We then have

$$K_\mathrm{t} = 2\cdot\frac{m}{2}\cdot\left(\frac{l\dot{\alpha}}{2\sqrt{2}}\right)^2 = \frac{ml^2\dot{\alpha}^2}{8}.$$

The rotational kinetic energy is

$$K_\mathrm{r} = 2\cdot\frac{I\dot{\alpha}^2}{2} = \frac{ml^2\dot{\alpha}^2}{12},$$

so the total is $K = K_\mathrm{t} + K_\mathrm{r} = (5/24)ml^2\dot{\alpha}^2$. Using energy conservation ($K + U = \mathrm{const}$) and differentiating with respect to time:

$$\frac{5}{12}ml^2\dot{\alpha}\ddot{\alpha} + 2kl^2\alpha\dot{\alpha} = 0 \quad\Rightarrow\quad \ddot{\alpha} + \frac{24k}{5m}\alpha = 0.$$

This corresponds to an angular frequency $\omega = \sqrt{24k/5m}$. The linear requency is then

$$\boxed{\nu = \frac{1}{2\pi}\sqrt{\frac{24k}{5m}}.}$$

Problem 3. Collision. A gamma-ray photon interacts with a stationary proton, producing a π^0 meson:

$$\gamma + p \to p + \pi^0$$

What is the minimum energy of the gamma-ray photon for this process to be possible? The rest masses of the proton and the pion are $m_p = 938\,\mathrm{MeV/c^2}$ and $m_\pi = 135\,\mathrm{MeV/c^2}$.

Solution. We'll work with $c = 1$ and restore the c's at the end. It's easiest to solve this problem with the 4-vector formalism. Let the photon have energy ε. It's massless, so its momentum is also ε. Its 4-vector is $(\varepsilon, \varepsilon)$, and so the net 4-vector of the photon and the stationary proton is $(\varepsilon + m_p, \varepsilon)$. From the conservation laws, this is the net 4-vector of the decay products as well.

Now we'll make use of the fact that the norm of the 4-vector is the same in all frames. The square of the norm is $(\varepsilon + m_p)^2 - \varepsilon^2$. This is also its value in the centre-of-mass frame of the decay proton and meson. But in that frame, this is just $(E_p + E_\pi)^2$, where E_p and E_π are the total energies of the proton and the meson (as the total momentum is zero by definition).

The minimum value of ε is achieved for minimum E_p and E_π. The best case scenario is if they both sit still in the centre-of-mass frame. That way, $E_p = m_p$, $E_\pi = m_\pi$, and momentum conservation is certainly satisfied. We then have

$$(\varepsilon + m_p)^2 - \varepsilon^2 = (m_p + m_\pi)^2 \quad\Rightarrow\quad \varepsilon = m_\pi c^2\left(1 + \frac{m_\pi}{2m_p}\right) = \boxed{145\,\mathrm{MeV}.}$$

Problem 4. Faraday homopolar generator. An ideal conducting disk of radius r_0 is placed in a constant uniform magnetic field with induction B, perpendicular to the disk. A resistor with resistance R is connected between the centre and the rim of the disk. A body of mass M is attached to a thread wound around the disk, as shown on Figure 4. The gravitational acceleration is g. The body begins to fall, and after some time the disk reaches its maximum angular velocity. Find this angular velocity ω and the current I flowing through the resistor.

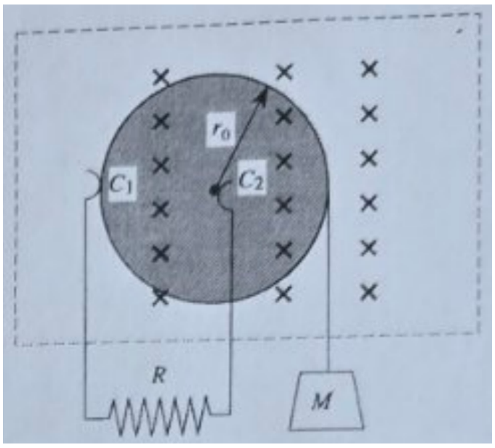

Figure 4

Solution. As the disk rotates with angular velocity ω, an EMF ε is induced between the centre and the rim. For a segment $\mathrm{d}r$ at distance r from the centre, the induced EMF is $\mathrm{d}\varepsilon = vB\,\mathrm{d}r = \omega r B\,\mathrm{d}r$. Integrating from 0 to r_0:

$$\varepsilon = \int_0^{r_0} \omega B r\,\mathrm{d}r = \frac{1}{2}\omega B r_0^2.$$

The current in the circuit is $I = \varepsilon/R = \omega B r_0^2/2R$. We want to find the braking torque due to this current. We don't know the exact current distribution, so let's split up the I that flows between C_1 and C_2 into thin strands that carry $\mathrm{d}I$ each and terminate at C_1 and C_2. Each strand consists of a series of current elements of length $\mathrm{d}\mathbf{l}$. The braking torque on one strand will then be

$$\mathrm{d}\boldsymbol{\tau}_b = \int_0^{r_0} \mathbf{r} \times (\mathrm{d}I\,(\mathrm{d}\mathbf{l} \times \mathbf{B})) = (\mathrm{d}I)\int_0^{r_0} \mathrm{d}\mathbf{l}(\cancel{\mathbf{r}\cdot\mathbf{B}}) - \mathbf{B}(\mathbf{r}\cdot\mathrm{d}\mathbf{l}) = -(\mathrm{d}I)\mathbf{B}\int_0^{r_0} r\,\mathrm{d}r = -\frac{1}{2}(\mathrm{d}I)\mathbf{B}r_0^2.$$

Denoting the unit vector into the page by $\hat{\mathbf{z}}$, the total braking torque is

$$\boldsymbol{\tau}_b = -\frac{1}{2}I\mathbf{B}r_0^2 = -\frac{\omega B^2 r_0^2}{4R}\hat{\mathbf{z}}.$$

At the maximum angular velocity, the driving torque from the falling mass $\boldsymbol{\tau}_d = Mgr_0\hat{\mathbf{z}}$ is balanced by the braking torque $\boldsymbol{\tau}_b$:

$$Mgr_0 = \frac{\omega B^2 r_0^4}{4R} \quad \Rightarrow \quad \boxed{\omega = \frac{4MgR}{B^2 r_0^3}.}$$

The current is

$$I = \frac{\omega B r_0^2}{2R} = \boxed{\frac{2Mg}{Br_0}.}$$

As an alternative solution, the current can be found by noting that at terminal velocity, the mechanical power $P_{\text{mech}} = Mgv = Mg\omega r_0$ is dissipated as Joule heat $P_{\text{heat}} = I\varepsilon$ entirely. After plugging in $\varepsilon = \omega B r_0^2/2$, we get the same result for I. The answer for ω follows immediately from $I = \varepsilon/R$.

Problem 5. AC bridge. An AC bridge (Figure 5) consists of a source of alternating voltage $V(t) = U_0 \cos(\omega t)$, two identical resistors, each with resistance R, a capacitance C, and an inductance L. An oscilloscope connected at the middle terminals of the bridge then measures a voltage $U(t) = U_1 \cos(\omega t + \varphi)$.

(a) For what relation between the parameters of the bridge do we get $U_1 = 0$?

(b) For arbitrary parameters of the bridge, there is some frequency at which the voltages $V(t)$ and $U(t)$ are in phase (i.e. $\varphi = 0$). Find this frequency ω_0. At this ω_0, what is the ratio U_1/U_0?

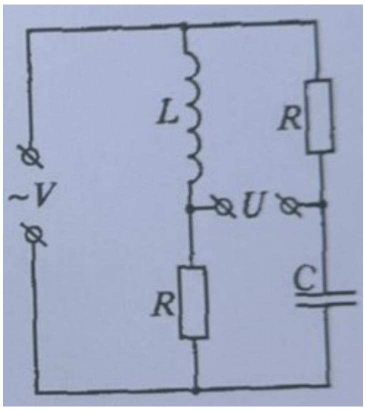

Figure 5

Solution. (a) We'll work with complex voltages. Choose the bottom terminal of the source as the ground. We'll determine the potentials at the terminals of the oscilloscope. At the left node A (between L and R):

$$V_A = U_0 \frac{R}{R + i\omega L} = \frac{U_0}{1 + i\omega \frac{L}{R}}.$$

At the right node B (between R and C):

$$V_B = U_0 \frac{1/(i\omega C)}{R + 1/(i\omega C)} = \frac{U_0}{1 + i\omega RC}.$$

The complex voltage at the output is

$$U = V_A - V_B = U_0 \left(\frac{1}{1 + j\omega \frac{L}{R}} - \frac{1}{1 + j\omega RC} \right).$$

For $U_1 = 0$, we require the two denominators to be equal:

$$\boxed{L/C = R^2.}$$

(b) To save on ink, define the time constants $\tau_1 = L/R$ and $\tau_2 = RC$. The formula for the output voltage simplifies to

$$U = U_0 \left(\frac{1}{1 + i\omega\tau_1} - \frac{1}{1 + i\omega\tau_2} \right) = U_0 \frac{i\omega(\tau_2 - \tau_1)}{(1 - \omega^2\tau_1\tau_2) + i\omega(\tau_1 + \tau_2)}.$$

For $U(t)$ and $V(t)$ to be in phase, the quantity U must be real (and positive, assuming $\tau_2 > \tau_1$). Since the numerator is purely imaginary, this is achieved when the denominator is also purely imaginary. Thus, the real part of the denominator must vanish:

$$1 - \omega_0^2 \tau_1 \tau_2 = 0 \quad \Rightarrow \quad \omega_0 = \frac{1}{\sqrt{\tau_1\tau_2}} = \boxed{\frac{1}{\sqrt{LC}}}.$$

At this frequency, the $i\omega$ terms in the numerator and the denominator cancel out, leaving us with the ratio

$$\boxed{\frac{U_1}{U_0} = \left|\frac{R^2C - L}{R^2C + L}\right|.}$$

We've inserted an absolute value because we don't have any information on how R, L, and C compare.

Problem 6. Thermal radiation. Consider a body in thermodynamic equilibrium with an evacuated cavity inside it. The energy density of the radiation inside the cavity is u, and the intensity of the thermal radiation incident on the cavity wall (energy per unit area per unit time) is Φ. Express Φ in terms of u.

Solution. Let's look at a small area element $\mathrm{d}A$ on the cavity wall. Consider the hemisphere of space that our area element gets photons from. We'll work with spherical polar coordinates, with ϕ being the azimuthal angle and θ being the polar angle ($\theta = 0$ along the normal). We'll look at the photons that arrive at $\mathrm{d}A$ from some piece of solid angle $\mathrm{d}\Omega$. The photons that could come from there are a fraction $\mathrm{d}\Omega/4\pi$ of all photons that bounce around in the cavity (the cavity is in thermodynamic equilibrium, so the radiation field inside is isotropic). The overall energy density u comes from just summing the energies of all photons in the cavity, so the photons from $\mathrm{d}\Omega$ can be thought to have an equivalent energy density $\mathrm{d}u' = u(\mathrm{d}\Omega/4\pi)$.

In a time interval $\mathrm{d}t$, the only radiation that will strike the area $\mathrm{d}A$ belongs to a cylinder of height $c\,\mathrm{d}t$ and base $\mathrm{d}A\cos\theta$. Then, the total energy $\mathrm{d}E$ incident on the surface element coming from $\mathrm{d}\Omega$ is

$$\mathrm{d}E = \mathrm{d}u'\,c\cos\theta\,\mathrm{d}A\,\mathrm{d}t = \left(u\frac{\mathrm{d}\Omega}{4\pi}\right)c\cos\theta\,\mathrm{d}A\,\mathrm{d}t = \frac{uc}{4\pi}\,\mathrm{d}A\,\mathrm{d}t\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

To find the total intensity Φ (the energy per unit area per unit time), we integrate this expression over all incoming directions ($\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$), and then divide by $\mathrm{d}A$ and $\mathrm{d}t$:

$$\Phi = \frac{1}{\mathrm{d}A\,\mathrm{d}t}\int \mathrm{d}E = \frac{uc}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta.$$

After evaluating the angular integrals,

$$\boxed{\Phi = \frac{uc}{4}.}$$

Problem 7. Pistons. In a long horizontal cylindrical tube there are two pistons of masses $M_1 = 2\,\mathrm{kg}$ and $M_2 = 1\,\mathrm{kg}$, which can move with virtually no friction. Between the pistons there is one mole of ideal gas (helium). We apply opposing forces $F_1 = 100\,\mathrm{N}$ and $F_2 = 50\,\mathrm{N}$ to the pistons, as shown on Figure 6. What is the equilibrium distance l between the pistons? The temperature of the gas T is constant and equal to $10\,\mathrm{K}$. The tube is in vacuum.

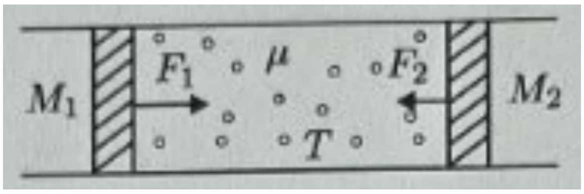

Figure 6

Solution. We're to realise that the system as a whole (consisting of both pistons and the gas) will accelerate to the right under the action of the net external force. At dynamic equilibrium, the distance l between the pistons remains constant, and both pistons accelerate at the same rate a. The mass of the helium gas is $4\,\mathrm{g/mol}\cdot 1\,\mathrm{mol} = 4\,\mathrm{g}$, which can be neglected compared to the mass of the pistons. Then, the acceleration is

$$a = \frac{F_1 - F_2}{M_1 + M_2}.$$

Let A be the cross-sectional area of the cylinder and P be the pressure of the gas (assume for now that it's the same everywhere). From Newton's second law for the left piston, we have

$$F_1 - PA = M_1 a \quad \Rightarrow \quad PA = \frac{M_1 F_2 + M_2 F_1}{M_1 + M_2}.$$

Now use the ideal gas law $PV = \nu RT$ for a volume $V = Al$:

$$l = \frac{\nu RT}{PA} = \nu RT \frac{M_1 + M_2}{M_1 F_2 + M_2 F_1} = \boxed{1.2\,\mathrm{m}.}$$

Finally, we need to verify that our assumption of constant pressure along the cylinder is reasonable. Since the gas is subjected to effective gravity a, a good estimate for the pressure difference at both ends is $\Delta P = \rho a l$, where ρ is the density of the gas. Let's find $\Delta P/P$:

$$\frac{\Delta P}{P} = \frac{\rho a l}{P} = \frac{\mu a l}{RT} = \frac{F_1 - F_2}{M_1 F_2 + M_2 F_1}(\mu\nu) = 10^{-3}.$$

All is good.

Problem 8. Photon in a cylinder. Consider a cylindrical chamber with a piston where both the cylinder base and the piston have a mirror coating. The chamber contains a single photon of frequency ω_0 whose momentum is directed perpendicularly to the piston. The piston begins to move slowly at constant speed until the volume of the container is reduced by a factor of k. Find the final frequency of the photon. Assume that the wavelength of the photon is much smaller than the dimensions of the chamber, and that the momentum of the photon is much smaller than that of the piston.

Solution. This is a classical problem, but we might as well do it using special relativity. Assume that the photon is just about to strike the piston when its frequency is ω. In the frame of the piston (which moves with velocity v towards the base), this frequency is

$$\omega' = \omega\sqrt{\frac{1 + v/c}{1 - v/c}}.$$

After reflection, the frequency in the piston's frame remains the same. However, in the lab frame it becomes

$$\omega'' = \omega'\sqrt{\frac{1 + v/c}{1 - v/c}} = \omega\frac{1 + v/c}{1 - v/c}.$$

In the classical regime, this implies a shift $\Delta\omega = (2v/c)\omega$. At the same time, the round-trip time between consecutive reflections of the photon off the piston is $\Delta t = 2L/c$. Therefore, the rate of change of the photon's frequency is:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} \approx \frac{\Delta\omega}{\Delta t} = \frac{v}{L}\omega$$

The length of the cylinder decreases at a rate of $\frac{\mathrm{d}L}{\mathrm{d}t} = -v$. Hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\omega \quad \Rightarrow \quad \frac{\mathrm{d}\omega}{\omega} = -\frac{\mathrm{d}L}{L}.$$

Integrating both sides yields $\omega L =$ constant. For a volume decrease of factor k, the length decreases by the same factor, so the final frequency is

$$\boxed{\omega = k\omega_0.}$$

There is also a simpler method. We recognise that the process at hand is adiabatic, so there exists an adiabatic invariant

$$I = \oint p\,\mathrm{d}x,$$

where the integral indicates a full "lap" of the photon in 1D position space, while p is the momentum of the photon. We substitute $p = \hbar\omega$, and we get $I = 2\hbar\omega L$. This is a constant, so again we get $\omega = k\omega_0$.

Problem 9. $\mathbf{CO_2}$ laser. The CO_2 molecule has a large number of closely spaced discrete energy levels, transitions between which can lead to the generation of laser radiation with a wavelength $\lambda \approx 10\,\mu\mathrm{m}$ and a spacing between the spectral lines $\Delta\lambda \approx 20\,\mathrm{nm}$. To achieve continuous tuning of the laser frequency, we increase the pressure in the cavity. The resulting broadening of the levels is governed by the intermolecular collisions, whose rate depends on the mean free time. The broadening causes the lines to merge into a single band.

Estimate the pressure P at temperature $T = 400\,\mathrm{K}$ for which this merging becomes possible. The collision cross-section between two molecules is $\sigma \approx 10^{-19}\,\mathrm{m}^2$.

Solution. The spectral lines will broaden because the excited CO_2 states have a finite lifetime τ due to the intermolecular collisions – in fact, τ is precisely the mean free time. The broadening of a line ΔE_b (i.e. the uncertainty in its energy) can be estimated using Heisenberg's uncertainty principle: $\Delta E_b \tau \sim \hbar$.

We'll assume that neighbouring lines merge when the broadening of each line becomes equal to the energy difference ΔE of the lines. That difference can be related to $\Delta\lambda$ after differentiating $\nu = c/\lambda$:

$$\Delta E = \hbar \cdot 2\pi\Delta\nu = \hbar \cdot \frac{2\pi c}{\lambda^2}\Delta\lambda.$$

We see that

$$\frac{1}{\tau} \approx \frac{2\pi c}{\lambda^2}\Delta\lambda.$$

The pressure of the gas can be found from $P = nk_BT$, where n is the number density of the CO_2 molecules. Our task is now to connect n to the mean free time in the gas.

The mean free path is $l = 1/\sigma n$, while the mean free time is defined as $\tau = l/v_T$, where $v_T = \sqrt{3RT/\mu}$ is the RMS speed of the molecules. Here $\mu = 0.044\,\mathrm{kg/mol}$ is the molar mass of CO_2. Substituting $n = 1/\sigma v_T \tau$ into the formula for the pressure, we obtain

$$P \approx \frac{2\pi c\Delta\lambda}{\sigma N_A \lambda^2}\sqrt{\frac{\mu RT}{3}} \approx \boxed{4 \times 10^7\,\mathrm{Pa.}}$$

Problem 10. Bose-Einstein condensate. An ideal monatomic gas of bosons (${}^4_2\mathrm{He}$) is cooled down at constant volume V and constant particle number N. As its temperature decreases, we reach a temperature T_0 below which the properties of the gas arise from the quantum properties of the bosons – i.e. their wavelike nature and their indistinguishability.

(a) Find T_0. The wavelike properties become significant when the de Broglie wavelength at the average thermal energy is approximately equal to the mean distance between the particles. Provide a numerical estimate for the number density $n = N/V$ if $T_0 = 4\,\mathrm{K}$.

At temperatures $T < T_0$ the particles of the gas can be separated into two groups, each encompassing a nonnegligible number of particles. The first group consists of N_0 particles at the lowest energy level ($\varepsilon = 0$), which do not take part in the thermal motion. The second group consists of N^* particles distributed across various energy levels (with $\varepsilon > 0$). These do take part in the thermal motion, and their number is given by

$$N^* = N\left(\frac{T}{T_0}\right)^{3/2}.$$

This is called a degenerate Bose gas.

(b) Find the heat capacity of the gas C_V when $T < T_0$.

(c) Find the pressure of the gas P when $T < T_0$. What is interesting about this result?

Apart from the thermodynamic variables T, V, and N, your results must include the Planck constant h, the Boltzmann constant k_B, and the mass of the helium atom, $m = 6.7 \times 10^{-27}\,\mathrm{kg}$.

Solution. This is a repeat of Short Exam 3 from the 2018 Bulgarian IPhO TST, with one subpart cut out. The solution can be found in the 2018 file – click here.

Experimental Exam

Problem 1. Measuring g with a ruler and a stopwatch.

Equipment:

1. Ruler with millimetre graduations and a plastic end-cap with a bearing fixed to it, for mounting the ruler on a rotation axis (Figure 7). The ruler may also be used for drawing.

   Note: The ruler and the end-cap form a single body that must not be disassembled. From here on, this composite body is just referred to as the ruler.

2. L-shaped steel bracket with an axis (bolt) for mounting the ruler.
3. Clamp for attaching the bracket to the table.
4. Stopwatch. The stopwatch is started by pressing the red button (on the right) and stopped by pressing it again. The display is reset using the black button on the left. If the stopwatch is off at the start of the experiment, hold down the red button until a zero reading appears on the display.
5. Two magnets of mass $m = 11.7\,\mathrm{g}$ each, to serve as weights. The magnets can be attached simultaneously on both sides of the ruler with their opposite poles facing each other.
6. Two sheets of graph paper.
7. Blank paper.

You may assume that the surface of the table is horizontal and its edges are perpendicular to each other.

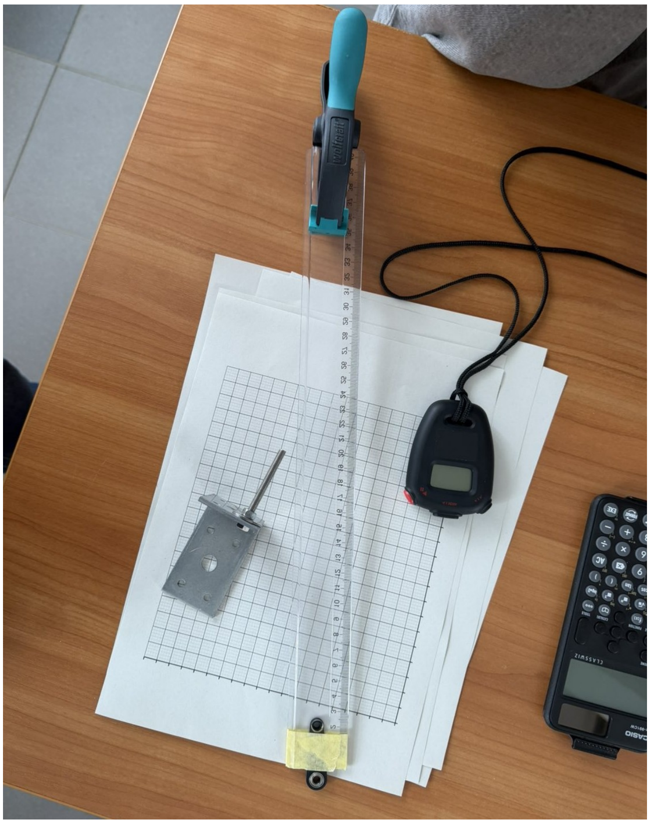

Figure 7

Tasks:

(a) Let us take the scale of the ruler as the x-axis. Provide theoretical justification and take measurements so as to determine the mass M of the ruler and the coordinate x_C of its centre of mass.

(b) Provide theoretical justification and take measurements to determine the moment of inertia I_C of the ruler about its centre of mass, as well as the value of the acceleration due to gravity g.

For both tasks, present your measurements and calculations using tables and graphs where appropriate.

Note:

1. If you are unable to determine the acceleration due to gravity, you can still find I_C after taking $g = 9.81\,\mathrm{m/s^2}$ as given. In this case, however, you will lose a significant portion of the marks for the second subpart.
2. Do not attempt to fabricate data when determining g! If you do this, you will get **zero** marks for the second subpart.

Problem 2. Diode and resistor circuit.

Equipment:
Circuit consisting of two identical diodes and a resistor (Figure 8; the diodes are connected in parallel with opposite orientations, and the resistor is in series with one of the diodes), rectifier which can supply either constant voltage or constant current, two multimeters, resistor substitution box (current not to exceed 100 mA), wires, graph paper.

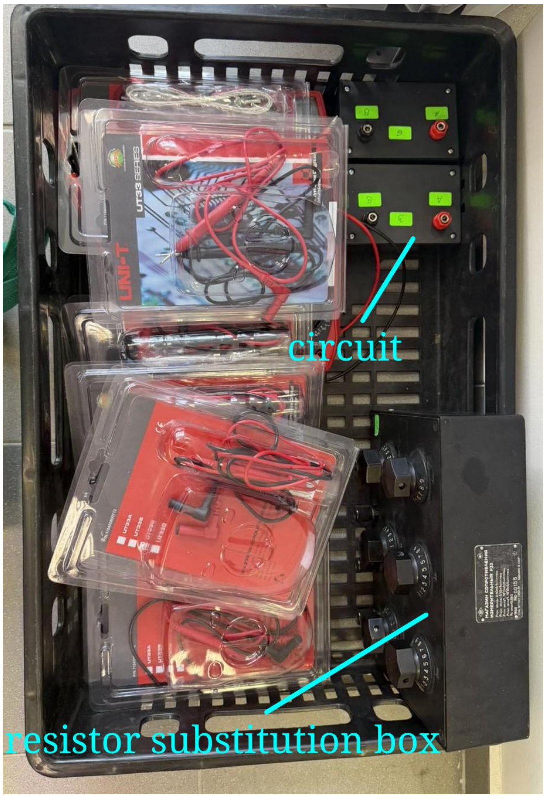

Figure 8

Task 1. Finding the resistance of the resistor R. **(7.0 pt)**
In this part of the problem you will measure the I-V curve of the circuit (without using the substitution box) for both positive and negative (i.e. with reversed polarity) voltages.

Note: Do not exceed a current of 3.0 A.

(a) Sketch the circuit that you have assembled.
(b) Write down the ranges that you use for the multimeters.
(c) Describe how R can be calculated from your measurements.
(d) How will you use the rectifier – to supply a constant voltage or to supply a constant current?

Note: The characteristics of the diodes have a strong dependence on temperature.

(e) Measure the I-V curve of the circuit as the voltage/current is raised. After you have reached the maximum voltage/current, wait until the open diode reaches its equilibrium temperature. Then, measure the I-V curve of the circuit as the voltage/current is lowered. Repeat this for voltages of the opposite polarity. Present your results in a table.

Note: A smell of hot plastic is to be expected, but in case of smoke, call the examiner immediately.

(f) Decide on the dataset that you will use for determining R. Choose between the values taken when raising the current/voltage and those taken when lowering the current/voltage.
(g) Plot a graph from which you can find R.
(h) Find R from the graph.
(i) Using the graph, find your error ΔR.

Task 2. Finding the reverse-bias saturation current of the diodes I_S. **(7.0 pt)**
The current I_S is the maximum current through a closed diode. The I-V curve of a diode can be modelled by the Shockley diode equation,

$$I = I_\mathrm{S}\left(e^{\frac{eU}{nk_BT}} - 1\right),$$

where e is the charge of the electron, k_B is the Boltzmann constant, T is the absolute temperature, and n is a number on the order of 1.

(a) Find an approximation of the formula above which can be used when measuring the forward I-V curve for voltages on the order of a few hundred mV at room temperature.
(b) Apply a voltage of such polarity that the diode with no resistor attached to it is open.
(c) Measure an appropriate part of the I-V curve for currents under 100 mA. Use the resistor substitution box if necessary. Present your results in a table.
(d) Plot your data in appropriate variables.
(e) Using the plot, find I_S and n.

Task 3. Using the data obtained so far, find the temperature of the diode T when a current $I_0 = 3\,\mathrm{A}$ flows through it. **(1.0 pt)**

Call the examiner if you suspect that a multimeter's battery has drained, that its fuse has blown, or in case of any other technical difficulties.